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A Technique for the Development of a Multiple Absolute Prediction Battery

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I. The Multiple Absolute Prediction Problem

In a previous report entitled "A Technique for the Development of a Differential Prediction Battery" (1) we have presented a technique for selecting from a battery of predictors a subset of specified size which will have the highest differential prediction efficiency for a given set of criterion variables. In subsequent references to this report it will be designated merely as "Differential Prediction." Differential prediction efficiency was defined in terms of the accuracy with which differences between all possible pairs of criterion measures could be predicted.

One may also be concerned with a technique for selecting from a battery of potential predictors that subset of specified size which will have the highest prediction efficiency for all the criterion variables irrespective of how well it differentiates among them. To distinguish between the two types of multiple prediction we shall call the former multiple differential prediction and the latter multiple absolute prediction. We may define the efficiency of multiple absolute prediction in terms of the accuracy with which all of the criteria are predicted, irrespective of how well the selected battery differentiates among the criteria.

In Differential Prediction we showed that an appropriate measure of the prediction efficiency of a selected battery is the difference between the average variance of the predicted criteria and their average covariance. In the case of multiple absolute prediction we may take as a measure of the prediction efficiency of the selected battery the sum of the variances of the predicted criteria irrespective of their average covariances. In the case of standard criterion measures such a measure can readily be shown to be the sum of the squares of the multiple correlations of the criteria with the predictors.

In both types of multiple prediction we consider only the case of standard criterion measures. However, it is a simple matter to weight the criterion variables differentially if such weights are available, so that the variances of the criterion variables would be proportional to the squares of the weights rather than all equal to unity.

As it turns out, in both differential and absolute prediction the end result is a set of predictions of success in each of the criterion activities. Furthermore, the weights to be used in both cases are the conventional least-square regression weights. The essential difference between the two cases is that a different set of predictors is sel-

[1] This research was carried out under Contract Nonr-477(08) between the University of Washington and the Office of Naval Research. The data used to illustrate the technique were provided by Professor August Dvorak. Computations were carried out by Robert Dear and William Meredith. Charlotte MacEwan assumed major editorial responsibility for the preparation of the manuscript. Much credit is due the typist, Eleanor Green. Supervision of both computational and editorial activities was provided by William Clemans. To each of these able contributors I am deeply grateful.

ected for the two types of prediction.

There may, however, be a great deal of overlap between the two sets of predictors, depending on the original group of potential predictors from which the subset is taken and the correlations among the criterion variables. Nevertheless, it was noted in the case of differential prediction that correlations among the criterion variables are not required by the selection procedure. Satisfactory estimates of the correlations of each criterion with the predictors are required. However, this is also true for the case of multiple absolute prediction.

In the selection procedure for the differential prediction battery one predictor at a time was selected. The first predictor selected was the one which according to the definition gave the highest index of differential prediction efficiency. The next predictor was the one which in combination with the first gave the highest index of differential prediction efficiency. Thus predictors are selected one at a time until the prespecified number has been selected. The assumption here is that this procedure of selecting a set of specified size yields an index of differential prediction efficiency sufficiently close for practical purposes to that of the particular set of specified size which, for the particular data, would yield the highest index of differential prediction efficiency.

Similarly, one test at a time is selected for the absolute prediction battery and the analogous assumption is made with reference to the index of absolute prediction efficiency of the selected battery.

II. The Computational Method

A. Selecting the Differential Predictors

Assume that we have given the matrix of intercorrelations of n potential predictors, and the matrix of correlations of the n predictors with N criterion variables. These are the same two sets of data used in Differential Prediction. As in that report we let

$_1g$ = the $(n \times n)$ matrix of intercorrelations of the n predictors,

$_1T$ = the $(n \times N)$ matrix of correlations of the n predictors with the N criteria.

Then, of course, each column of $_1T$ represents a criterion variable and each row a predictor variable. The following steps in the predictor selection procedure correspond to those illustrated by the numerical example in Part III, and reference to the example should clarify these steps. For the example, the same data are used in this report as in Differential Prediction. (It will be noted that in the example, the $_1T'$ matrix is used rather than $_1T$. In fact, the transpose of $_iT$ is used throughout in the numerical computations as a more convenient form for accommodating all the matrix elements and the relevant computations on one worksheet. This form is especially convenient if the number of criterion variables is large. Consequently operations defined in the current sections for *rows* of the T matrices have been performed, in the example, upon *columns*.)

1. We start by computing the sums of squares of each row of the $_1T$ matrix and call these values for the kth row $_1V_k$. The equation for $_1V_k$ is simply

$$_1V_k = \Sigma_1 T_k^2. \qquad [1]$$

In matrix notation Equation 1 is given by

$$_1V_k = {}_1T_{k.}' \, {}_1T_{k.}, \qquad [2]$$

where $_1T_{k.}'$ is the kth row vector of $_1T$.

We take the predictor variable corresponding to the largest $_1V_k$ as the first predictor in the subset to be selected and designate this as predictor a.

2. Next we compute a new matrix $_2g$ from the $_1g$ matrix and its ath row or column by the formula

$$ {}_2g_{kp} = {}_1g_{kp} - {}_1g_{ak}\ {}_1g_{ap}. \qquad [3] $$

This step is precisely the same as for Differential Prediction except that the ath variable will not in general be the same for both types of selection. Equation 3 means that the element in the kth row and pth column of ${}_2g$ is obtained by subtracting from the corresponding element in ${}_1g$ the product of the elements in the kth row and ath column and the pth row and ath column of ${}_1g$. As in the case of Differential Prediction the ath row and column of ${}_2g$ will have all zero elements. In matrix notation Equation 3 is

$$ {}_2g = {}_1g - {}_1g_{.a}\ {}_1g_{.a}', \qquad [4] $$

where ${}_1g_{.a}$ is the ath column vector of ${}_1g$ and ${}_1g_{.a}'$ is its transpose.

3. Compute a ${}_2T$ matrix from the ${}_1T$ matrix and the ath row of the ${}_1g$ matrix by means of the scalar formula,

$$ {}_2T_{kp} = {}_1T_{kp} - {}_1T_{ak}\ {}_1g_{ap}. \qquad [5] $$

This step is the same as step 3 in Differential Prediction except that the ath variable will not in general be the same. Equation 5 means that the element in the kth row and pth column of ${}_2T$ is obtained by subtracting from the corresponding element of ${}_1T$ the product of the element in the ath row and kth column of ${}_1T$ by the element in the ath row and pth column of ${}_1g$. As in Differential Prediction the ath row of ${}_2T$ will have all zero elements. In matrix notation Equation 5 is

$$ {}_2T = {}_1T - {}_1g_{.a}\ {}_1T_{a.}', \qquad [6] $$

where ${}_1T_{a.}'$ is the ath row vector of ${}_1T$ and ${}_1g_{.a}$ is the ath column vector of ${}_1g$. Since ${}_1g$ is symmetrical we could, of course, take either its ath column vector or the transpose of its ath row vector.

4. Compute the sum of squares for each row of ${}_2T$ obtained in step 3 by means of the scalar formula

$$ {}_2V_k = \Sigma {}_2T_k{}^2. \qquad [7] $$

In matrix notation Equation 7 is

$$ {}_2V_k = {}_2T_{k.}'\ {}_2T_{k.}, \qquad [8] $$

where the notation is analogous to that defined for Equation 2. It should be noted that step 4 is somewhat simpler than the corresponding step for Differential Prediction. For the latter case we compute N times the variance of each row instead of simply the sum of squares.

5. Next divide each of the ${}_2V_k$ values computed in step 4 by the corresponding diagonal element of ${}_2g$ obtained in step 2 and designate the kth ratio as ${}_2\Delta_k$. In scalar notation this is

$$ {}_2\Delta_k = \frac{{}_2V_k}{{}_2g_{kk}}. \qquad [9] $$

In matrix notation Equation 9 may be written

$$ D_{2^\Delta} = D_{2^\varphi}{}^{-1} D_{2^V}, \qquad [10] $$

where

D_{2^Δ} is a diagonal matrix of the ${}_2\Delta_k$,

D_{2^φ} is a diagonal matrix of the diagonal elements of ${}_2g$,

D_{2^V} is a diagonal matrix of the ${}_2V_k$.

Step 5 is analogous to the same step for Differential Prediction. The denominator terms for the two steps are in both cases the diagonal elements of the ${}_2g$ matrix although as previously noted the ${}_2g$ matrix will not in general be the same for both procedures. The numerator terms, however, are different since they come from step 4 which is different for the two procedures.

The predictor variable corresponding to the largest ${}_2\Delta_k$ is the second selected predictor. This we designate as predictor b.

6. Divide each element of the bth row of ${}_2g$ by its bth element and designate the ratio by ${}_2G_{bk}$. In scalar notation this is

$$ {}_2G_{bk} = \frac{{}_2g_{bk}}{{}_2g_{bb}}, \qquad [11] $$

or in matrix notation

$$ {}_2G_b' = \frac{{}_2g_{b\cdot}'}{{}_2g_{bb}}. \qquad [12] $$

This step is precisely the same as for step 6 in Differential Prediction except that, of course, the bth predictor will not in general be the same for both cases.

7. Compute a new matrix ${}_3g$ from the ${}_2g$ matrix of step 2, its bth column and ${}_2G_b$ of step 6 by the scalar formula

$$ {}_3g_{kp} = {}_2g_{kp} - {}_2g_{bk}\,{}_2G_{bp}. \qquad [13] $$

This step is the same as in Differential Prediction and means that the element in the kth row and pth column of ${}_3g$ is obtained by subtracting from the corresponding element of ${}_2g$ the product of the element in the bth row and kth column of ${}_2g$ by the pth element of ${}_2G_b$. As in Differential Prediction, both the ath and bth rows and columns of ${}_3g$ will have all zero elements. In matrix notation Equation 13 is

$$ {}_3g = {}_2g - {}_2g_b\,{}_2G_b'. \qquad [14] $$

8. Compute a new matrix ${}_3T$ from the ${}_2T$ matrix, its bth row and the vector from step 6 by means of the scalar equation

$$ {}_3T_{kp} = {}_2T_{kp} - {}_2T_{bk}\,{}_2G_{bp}. \qquad [15] $$

This step is also the same as for Differential Prediction and means that the element in the kth row and pth column of ${}_3T$ is obtained by subtracting from the corresponding element in ${}_2T$ the product of the element in the bth row and kth column of ${}_2T$ by the pth element of the ${}_2G_b$ vector.

As in Differential Prediction both the ath and bth rows of ${}_3T$ will have all zero elements.

In matrix notation Equation 15 may be written

$$ {}_3T = {}_2T - {}_2G_b\,{}_2T_b'. \qquad [16] $$

9. Compute the sums of squares for each row of ${}_3T$ obtained in step 8 by means of the scalar equation

$$ {}_3V_k = \Sigma_3 T_k^2, \qquad [17] $$

which in matrix notation is

$$ {}_3V_k = {}_3T_{k\cdot}'\,{}_3T_{k\cdot}. \qquad [18] $$

10. Divide each ${}_3V_k$ computed in step 9 by the corresponding diagonal element of ${}_3g$ obtained in step 7 and designate the kth ratio as ${}_3\Delta_k$. In scalar notation this is

$$ {}_3\Delta_k = \frac{{}_3V_k}{{}_3g_{kk}}, \qquad [19] $$

or in matrix notation

$$ D_3\Delta = D_{3g}^{-1}\, D_3V, \qquad [20] $$

where the notation is as defined for Equation 10. The variable corresponding to the highest ${}_3\Delta_k$ is the third or kth predictor to be selected.

11. Repeat steps 6 through 10 to get the fourth and subsequent predictors of the subset. If we let w be the ith selected predictor, then the general equations representing the five steps required for selecting all predictors after the first two are given in scalar notation by the following five equations respectively.

(1) $$ {}_iG_{wk} = \frac{{}_ig_{wk}}{{}_ig_{ww}}, \qquad [21] $$

(2) $$ {}_{(i+1)}g_{kp} = {}_ig_{kp} - {}_ig_{wk}\,{}_iG_{wp}, \qquad [22] $$

(3) $$ {}_{(i+1)}T_{kp} = {}_iT_{kp} - {}_iT_{wk}\,{}_iG_{wp}, \qquad [23] $$

(4) $$ {}_{(i+1)}V_k = \Sigma_{(i+1)}T_k^2, \qquad [24] $$

(5) $$ {}_{(i+1)}\Delta_k = \frac{{}_{(i+1)}V_k}{{}_{(i+1)}g_{kk}}. \qquad [25] $$

The first three of these steps, as previously indicated, are the same as for Differential Prediction.

In matrix notation the corresponding steps are

(1) $$ {}_iG_w' = \frac{{}_ig_{w\cdot}'}{{}_ig_{ww}}, \qquad [26] $$

$$(2) \qquad {}_{(i+1)}g = {}_{i}g - {}_{i}g_{.w}\,{}_{i}G_{w}', \qquad [27]$$

$$(3) \qquad {}_{(i+1)}T = {}_{i}T - {}_{i}G_{w}\,{}_{i}T_{w.}', \qquad [28]$$

$$(4) \qquad {}_{(i+1)}V_{k} = {}_{(i+1)}T_{k.}'\,{}_{(i+1)}T_{k.}, \qquad [29]$$

$$(5) \qquad D_{(i+1)}{}^{\Delta} = D_{(i+1)}{}^{v-1}D_{(i+1)}{}^{V}. \qquad [30]$$

As in the case of Differential Prediction each new ${}_{i}T$ matrix has one more vanishing row than the preceding one and each new ${}_{i}g$ matrix has one more vanishing row and column than the preceding one.

Steps 6 through 10 may be repeated until the desired number of predictors has been selected. This number may be determined on the basis of administrative considerations such as the total amount of testing time available or limitations on the facilities available for processing the data. The selection procedure may also be terminated if it is believed that the Δ values are too small to justify adding additional variables. It should not be too difficult to develop tests to determine whether a given Δ differs significantly from zero. However, it is beyond the scope of this report to do so. It is proved in Part IV that the sum of all the Δ's is given by

$$\lambda = {}_{1}\Delta_{a} + {}_{2}\Delta_{b} + \cdots + {}_{i}\Delta_{w} = \Sigma_{i}C_{kk}, \qquad [31]$$

where ${}_{i}C_{kk}$ is the variance of the predicted criterion measure k or simply the square of the multiple correlation of criterion k with the i selected predictors. The sum of these variances is what we have defined as the index of multiple absolute prediction efficiency.

B. Solving for the Regression Vectors

For any specified set of predictors and criterion variables it has been proved in Differential Prediction that the least-square regression vectors will yield the highest index of differential prediction efficiency. In this report it is proved that the least-square regression vectors will also yield the highest index of absolute prediction efficiency. Therefore, for any specified set of predictors and criteria the same set of regression vectors will be used for both types of prediction. Assuming, however, a specified set of potential predictors and a set of criterion measures, the procedures for selecting a subset of predictors of specified size will not in general yield the same subset for both types of prediction. However, once a subset of predictors has been selected for either type of prediction, the procedures for calculating the regression vectors and multiple correlations are the same for both cases. Instead of repeating the procedures in detail we shall therefore give only a brief summary. The steps for computing the regression vectors are as follows:

1. Make up a matrix L from certain of the elements in the G vectors indicated by Equations 21 and 26. All elements in and below the diagonal of the L matrix are zero. The elements in the first row are taken, with opposite sign, from the ath row of the ${}_{1}g$ matrix and are, in order of selection, the elements corresponding to all but the first selected variable. The second row consists of those elements, with signs changed, from the ${}_{2}G$ vector (see Equations 11 and 12) corresponding, in the order of selection, to all but the first two selected predictors. Similarly for subsequent rows the elements are taken in order of selection, with signs changed, from the corresponding ${}_{i}G$ vector. Assuming four selected predictors the L matrix may be indicated by

$$L = \begin{pmatrix} 0 & L_{ab} & L_{ac} & L_{ad} \\ 0 & 0 & L_{bc} & L_{bd} \\ 0 & 0 & 0 & L_{cd} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad [32]$$

2. Next we make up a matrix the same order as L in Equation 32, which we call F', with all zero elements above the di-

agonal, thus

$$F' = \begin{bmatrix} F_{aa} & 0 & 0 & 0 \\ F_{ab} & F_{bb} & 0 & 0 \\ F_{ac} & F_{bc} & F_{cc} & 0 \\ F_{ad} & F_{bd} & F_{cd} & F_{dd} \end{bmatrix}. \quad [33]$$

The diagonal elements of F' are the reciprocals of the diagonal elements taken from the $_ig$ matrices. These are,

$$\left.\begin{aligned} F_{aa} &= \frac{1}{{}_1g_{aa}} = 1.00 \\ F_{bb} &= \frac{1}{{}_2g_{bb}} \\ F_{cc} &= \frac{1}{{}_3g_{cc}} \\ &\text{etc.} \end{aligned}\right\}. \quad [34]$$

The other elements of F' are given by the following sets of equations:

$$F_{ab} = L_{ab}F_{bb}, \quad [35]$$

$$\left.\begin{aligned} F_{bc} &= L_{bc}F_{cc} \\ F_{ac} &= L_{ab}F_{bc} + L_{ac}F_{cc} \end{aligned}\right\}, \quad [36]$$

$$\left.\begin{aligned} F_{cd} &= L_{cd}F_{dd} \\ F_{bd} &= L_{bc}F_{cd} + L_{bd}F_{dd} \\ F_{ad} &= L_{ab}F_{bd} + L_{ac}F_{cd} + L_{ad}F_{dd} \end{aligned}\right\}. \quad [37]$$

In matrix notation, the above solution for F', known as the "back solution," may be expressed as the product of two supermatrices:

$$(F' \mid D_{\sigma}^{-1}) \begin{pmatrix} L' \\ I \end{pmatrix} = F'.$$

The method of solving for the elements of F' is given, in brief, in Part II of Differential Prediction.

3. To check the elements of F' in Equation 37 we first consider a matrix $g_{(i)}$ of intercorrelations of the selected predictors arranged in the order of selection. We indicate the sums of these columns by Σg_{ai}, Σg_{bi}, Σg_{ci}, and Σg_{di}. Then the checks for the rows of F' are given by

$$\left.\begin{aligned} F_{ab}\Sigma g_{ai} + F_{bb}\Sigma g_{bi} &= 1 - \Sigma L_{bi} \\ F_{ac}\Sigma g_{ai} + F_{bc}\Sigma g_{bi} + F_{cc}\Sigma g_{ci} &= 1 - \Sigma L_{ci} \\ F_{ad}\Sigma g_{ai} + F_{bd}\Sigma g_{bi} + F_{cd}\Sigma g_{ci} + F_{dd}\Sigma g_{di} &= 1 - \Sigma L_{di} \end{aligned}\right\}. \quad [38]$$

Actually it is not necessary to make up a table of the intercorrelations of selected predictors. The required sums may be readily computed directly from the $_1g$ matrix. If the final equation in Equation 38 checks, this is also a check on the correlation sums.

In matrix notation the checks indicated by Equation 38 are given by

$$F'(g_{(i)}{}^1) = 1 - L1. \quad [39]$$

4. Next we make up a matrix t such that the first row is the row of $_1T$ corresponding to the first selected predictor. The second row is the row of $_2T$ corresponding to the second selected predictor and so on. We have therefore,

$$t = \begin{pmatrix} t_{a1}\, t_{a2} \cdots t_{aN} \\ t_{b1}\, t_{b2} \cdots t_{bN} \\ t_{c1}\, t_{c2} \cdots t_{cN} \\ t_{d1}\, t_{d2} \cdots t_{dN} \end{pmatrix}. \quad [40]$$

5. To get the regression vector $\beta_{.k}$ for the kth criterion we use the F matrix and the kth column of t in Equation 40. In scalar notation we have

$$\left.\begin{aligned} \beta_{ak} &= F_{aa}t_{ak} + F_{ab}t_{bk} + F_{ac}t_{ck} \cdots \\ \beta_{bk} &= \qquad F_{bb}t_{bk} + F_{bc}t_{ck} \cdots \\ &\text{etc.} \end{aligned}\right\}. \quad [41]$$

If we let β be the matrix of regression vectors in which the rows represent selected predictors and the columns criteria we can write Equation 41 in matrix notation as

$$\beta = Ft. \quad [42]$$

C. *Solving for the Multiple Correlations*

The solution for the multiple correlations proceeds exactly as in the case of Differential Prediction except, of course, that the selected predictors will not in general be the same. The check on the computations has, however, been simplified.

1. First we make up a matrix from the rows of the ${}_1T$ matrix corresponding to the selected predictors. The rows are written in the order in which the variables were selected and the matrix is designated ${}_1T_{(i)}$.

2. We then compute the squares of the multiple correlations by means of the conventional scalar formula,

$$R^2=\sum_i r_{ic}\beta_{ic}, \qquad [43]$$

where the r_{ic} are the correlations of the selected predictors with a given criterion and the β_{ic} are the corresponding beta weights. If we now consider only the i selected predictors and let

${}_1T_{.k}$ be the kth column of ${}_1T_{(i)}$,

$\beta_{.k}$ be the column vector of beta weights for the criterion k, and

R_k be the multiple correlation of criterion k with the predictors,

we have in matrix notation

$$R_k^2={}_1T_{.k}'\beta_{.k}. \qquad [44]$$

3. The R_k^2 are checked by summing them and seeing that the total is the same as the sum of the ${}_i\Delta$ given by Equation 31. In scalar notation the check is given by

$$\Sigma R_k^2=\lambda_i. \qquad [45]$$

4. If desired, the square roots of the R_k^2 computed in step 2 may be calculated to give simply the multiple correlations.

III. A Numerical Illustration

A. *The Data*

In this part we shall illustrate the methods outlined in Part II by a numerical example. The same data will be used as in the Differential Prediction report so that the results of the two methods may be compared.

The sample consists of 2,243 entering freshmen at the University of Washington for the fall of 1947. There are eight predictor variables as follows: the mean of high school grades in (1) English, (2) mathematics, (3) foreign language, (4) social science, (5) natural science, (6) electives, and the Q and L scores on the American Council on Education Examination.

The criterion measures are the grade point averages for ten different college course areas. These subjects, together with the number of cases for each course are: anthropology (603), chemistry (752), economics (1125), English (2005), foreign language (619), geology (475), history (583), mathematics (848), psychology (1255), and zoology (469).

B. *Computations for Selecting the Predictors*

The illustrations of the computational procedures will follow the same numbering as the steps in Part II, Section A. Table 1 gives the same matrix of intercorrelations of predictor variables as was used in the Differential Prediction study. As in that study the predictor selection process is carried out until all eight predictors have been selected, although the multiple regression vectors and the multiple correlations are based on the first four selected predictors just as in the case of the former study. Table 1 is the one which we have designated as ${}_1g$ in Part II. For checking purposes the matrix has been summed by rows and columns and a grand total is entered in the lower right-hand corner.

Table 2 is the same matrix of validity coefficients used in the Differential Pre-

TABLE 1
THE $_1g$ MATRIX OF PREDICTOR INTERCORRELATIONS

	1	2	3	4	5	6	7	8	Σ
1	1.0000	.5647	.6437	.7193	.5918	.4719	.1144	.3470	4.4528
2	.5647	1.0000	.5893	.5366	.5801	.3737	.3496	.2545	4.2485
3	.6437	.5893	1.0000	.5679	.5490	.4003	.1458	.2983	4.1943
4	.7193	.5366	.5679	1.0000	.5936	.4051	.1474	.3230	4.2929
5	.5918	.5801	.5490	.5936	1.0000	.4372	.2000	.3185	4.2702
6	.4719	.3737	.4003	.4051	.4372	1.0000	.1405	.1168	3.3455
7	.1144	.3496	.1458	.1474	.2000	.1405	1.0000	.4160	2.5137
8	.3470	.2545	.2983	.3230	.3185	.1168	.4160	1.0000	3.0741
Σ	4.4528	4.2485	4.1943	4.2929	4.2702	3.3455	2.5137	3.0741	30.3920
Check									30.3920

diction report. It is the transposed form of what we have called $_1T$ in Part II.

1. The first row at the bottom of Table 2 consists of column summations. The *a* row consists of sums of squares of column elements. These are given by Equation 1 of Part II. The largest value in the *a* row is 1.6671 for variable 5. This variable, therefore, becomes the first selected predictor, namely, variable *a*.

2. Next we compute the $_2g$ matrix shown in Table 3. This is computed by means of Equation 3 or 4 of Part II. Since the table is symmetrical, it is necessary to compute only the elements in and below the diagonal; the summation row of $_1g$ is included in the computations and the resulting values are entered in the "check" row at the bottom of $_2g$. The summation row of $_2g$ is obtained by adding across a row through the diagonal and then down the column, exclusive of the check item.

We may illustrate the use of equations 3 and 4 of Part II for getting the first column of $_2g$ as follows:

	$_1g_1$		$_1g_5$		$_1g_{51}$		$_2g_1$	
1	1.0000	−	.5918	×	.5918	=	.6498	
2	.5647	−	.5801	×	.5918	=	.2214	
3	.6437	−	.5490	×	.5918	=	.3188	
4	.7193	−	.5936	×	.5918	=	.3680	
5	.5918	−	1.0000	×	.5918	=	.0000	
6	.4719	−	.4372	×	.5918	=	.2132	
7	.1144	−	.2000	×	.5918	=	−.0040	
8	.3470	−	.3185	×	.5918	=	.1585	
Σ	4.4528	−	4.2702	×	.5918	=	1.9257	Check
							1.9257	Σ

TABLE 2
THE $_1T'$ MATRIX OF VALIDITY COEFFICIENTS

Crit.	Predictors								Total
	1	2	3	4	5	6	7	8	
1	.4378	.3357	.4188	.4793	.3950	.2053	.2257	.3764	
2	.3488	.4716	.3737	.3876	.4966	.2855	.2958	.2891	
3	.4028	.4200	.3645	.4490	.4329	.2166	.2727	.3721	
4	.4865	.3674	.4045	.4410	.4184	.2848	.2055	.4427	
5	.3968	.3852	.4400	.3793	.3855	.2794	.1288	.2419	
6	.3383	.3349	.3216	.3701	.3836	.1972	.1963	.3185	
7	.3440	.2335	.3208	.3851	.2870	.1572	.0952	.2899	
8	.3139	.4257	.3796	.2928	.4156	.2673	.2998	.2765	
9	.3555	.3029	.3228	.3969	.3799	.1992	.1978	.3387	
10	.4071	.4490	.3531	.4119	.4656	.2375	.2633	.4210	
Σ	3.8315	3.7259	3.6994	3.9930	4.0601	2.3300	2.1809	3.3168	27.1376
a	1.4931	1.4361	1.3844	1.6181	1.6771	0.5609	0.5176	1.1470	

TABLE 3

THE $_2g$ MATRIX

	1	2	3	4	5	6	7	8
1	.6498							
2	.2214	.6635						
3	.3188	.2708	.6986					
4	.3680	.1923	.2420	.6476				
5	0	0	0	0	0			
6	.2132	.1201	.1603	.1456	0	.8089		
7	−.0040	.2336	.0360	.0287	0	.0531	.9600	
8	.1585	.0697	.1234	.1339	0	−.0224	.3523	.8986
Check	1.9257	1.7714	1.8500	1.7581	0	1.4786	1.6597	1.7140
Σ	1.9257	1.7714	1.8499	1.7581	0	1.4788	1.6597	1.7140

To get the second column of $_2g$ we use the second column of $_1g$ and the second element of the fifth column of $_1g$. The remaining columns are obtained by using the appropriate columns of $_1g$ and the corresponding elements of the fifth column of $_1g$.

3. The $_2T$ matrix is computed by means of Equation 5 or 6 of Part II from the $_1T$ matrix, its ath or fifth row and the fifth column (or row) of the $_1g$ matrix. Table 4 is the $_2T$ matrix in transposed form. The use of Equation 5 or 6 of Part II may be illustrated for getting the first column of $_2T'$ as follows:

	$_1T_1'$		$_1T_5'$		$_1g_{51}$		$_2T_1'$
1	.4378	−	.3950	×	.5918	=	.2040
2	.3488	−	.4066	×	.5918	=	.0549
3	.4028	−	.4329	×	.5918	=	.1466
4	.4865	−	.4184	×	.5918	=	.2389
5	.3968	−	.3855	×	.5918	=	.1687
6	.3383	−	.3836	×	.5918	=	.1113
7	.3440	−	.2870	×	.5918	=	.1742
8	.3139	−	.4156	×	.5918	=	.0679
9	.3555	−	.3799	×	.5918	=	.1707
10	.4071	−	.4656	×	.5918	=	.1316
Σ	3.8315	−	4.0601	×	.5918	=	1.4288 Check
							1.4287 Σ

The remaining columns of $_2T'$ are obtained in the same way by using the appropriate columns of the $_1T'$ matrix and the appropriate element from the fifth row of $_1g$.

TABLE 4

THE $_2T'$ MATRIX

	1	2	3	4	5	6	7	8
1	.2040	.1066	.2019	.2448	0	.0326	.1467	.2506
2	.0549	.1835	.1011	.0928	0	.0684	.1965	.1309
3	.1466	.1689	.1268	.1920	0	.0273	.1861	.2342
4	.2389	.1247	.1748	.1926	0	.1019	.1218	.3094
5	.1687	.1616	.2284	.1505	0	.1109	.0517	.1191
6	.1113	.1124	.1110	.1424	0	.0295	.1196	.1963
7	.1742	.0670	.1632	.2147	0	.0317	.0378	.1985
8	.0679	.1846	.1514	.0460	0	.0856	.2167	.0941
9	.1307	.0825	.1142	.1714	0	.0331	.1218	.2177
10	.1316	.1789	.0975	.1355	0	.0339	.1702	.2727
Check	1.4287	1.3706	1.4704	1.5829	0	0.5549	1.3689	2.0237
Σ	1.4288	1.3707	1.4703	1.5827	0	0.5549	1.3689	2.0235
a	.2334	.2052	.2342	.2814	0	.0406	.2188	.4532
b	.3592	.3093	.3552	.4345	0	.0502	.2279	.5043

TABLE 5

MATRIX OF ${}_1G_w'$ VECTORS WITH RECIPROCALS OF THE DIAGONAL ELEMENTS FROM THE CORRESPONDING ${}_1g_w$ VECTORS

	$1/{}_1g_{ww}$	1	2	3	4	5	6	7	8	Check	Σ
5	1.00000	.5918	.5801	.5490	.5936	1.0000	.4372	.2000	.3185	4.2702	4.2702
8	1.11284	.1764	.0776	.1373	.1490	0	−.0249	.3921	1.0000	1.9074	1.9075
4	1.59337	.5488	.2898	.3563	1.0000	0	.2373	−.0379	0	2.3942	2.3943
2	1.65180	.1805	1.0000	.3244	0	0	.1298	.3522	0	1.9870	1.9869
3	1.85770	.2578	0	1.0000	0	0	.1575	−.1358	0	1.2796	1.2796
7	1.35870	−.0988	0	0	0	0	.0697	1.0000	0	.9709	.9709
1	2.70197	1.0000	0	0	0	0	.2824	0	0	1.2824	1.2824
6	1.39606	0	0	0	0	0	1.0000	0	0	1.0000	1.0000

4. The rows at the bottom of the ${}_2T'$ matrix are not quite the same as for the ${}_1T'$ matrix. The first row at the bottom is labeled "check" and is obtained by applying Equation 5 of Part II to the Σ row of ${}_1T'$. The second or Σ row consists simply of column sums, exclusive of the check entry and should, within rounding errors, be the same as the check row immediately above it. Row *a* is obtained in precisely the same way as row *a* of ${}_1T'$. It consists of the sums of squares of column elements given by ${}_2V_k$ of Equation 7 or 8 of Part II.

5. Row *b* is obtained by dividing each element in row *a* by the corresponding diagonal element of the ${}_2g$ matrix. These are the ${}_2\Delta_k$ values given by Equation 9 or 10 of Part II. For example, the first element in row *b* is

$$ {}_2\Delta_1 = \frac{{}_2V_1}{{}_2g_{11}} = \frac{.2334}{.6498} = .3592. $$

It may be noted that in the case of ${}_1T'$ we would also have a *b* row obtained by dividing each element in its *a* row by the corresponding diagonal element of the ${}_1g$ matrix. However, the diagonal elements of the ${}_1g$ matrix are all unity. Therefore, the *b* row for the ${}_1T'$ matrix would be identical to its *a* row.

The highest value in row *b* of Table 4 is .5043 for variable 8. Therefore, the second selected predictor *b* is variable 8.

Using Equation 31 of Part II we have as our index of absolute prediction efficiency for the first two selected variables

$$ \lambda_2 = {}_1\Delta_8 + {}_2\Delta_8 = 1.6771 + .5043 = 2.1814. $$

6. Each element of the *b*th or eighth row of ${}_2g$ in Table 3 is now divided by the eighth diagonal element, as described before, to get the ${}_2G$ values given by Equation 11 or 12 of Part II and the results are entered in the second row of Table 5. The first row is simply the *a*th or second row of the ${}_1g$ matrix of Table 1. Since the reciprocal of the eighth diagonal element of the ${}_2g$ matrix is required in subsequent computations, it is computed first and entered to the left of the second row of the *G* matrix. Then each element of the eighth row of ${}_2g$ is multiplied by it to get the other entries in the row. The check sum and the eighth element are both included. The eighth element in the second row of Table 5 should be unity. This checks the computation of the reciprocal. The total of the entries exclusive of the reciprocal and the check is entered at the extreme right of the row. This should be the same as the check entry immediately to its left.

7. The ${}_3g$ matrix is now computed from the ${}_2g$ matrix of step 2, its *b*th or eighth column and the ${}_3G_8'$ vector of step 6. The procedure is exactly the same as for step 2 except that the constant multiplier for a given column is the corresponding element of the ${}_3G_8'$ vector in the second row of

TABLE 6

THE $_3g$ MATRIX

	1	2	3	4	5	6	7	8
1	.6218							
2	.2091	.6581						
3	.2970	.2612	.6817					
4	.3444	.1819	.2236	.6276				
5	0	0	0	0	0			
6	.2172	.1218	.1634	.1489	0	.8083		
7	−.0661	.2063	−.0124	−.0238	0	.0619	.8219	
8	0	0	0	0	0	0	0	0
Check	1.6234	1.6384	1.6146	1.5027	0	1.5215	.9876	0
Σ	1.6234	1.6384	1.6145	1.5026	0	1.5215	.9878	0

Table 5. The $_3g$ matrix is given in Table 6.

8. Next the $_3T'$ matrix is computed from the $_2T'$ matrix of step 3, its bth or eighth column and the $_2G_8'$ vector of step 6. The procedure is the same as for step 3 except that the constant multiplier for a given column is taken from the corresponding element of the $_2G_8'$ vector in the second row of Table 5. Table 7 gives the $_3T'$ matrix.

9. Row a of the $_3T'$ matrix in Table 7 is simply the sums of squares of column elements as in the case of $_1T'$ and $_2T'$ in steps 1 and 4.

10. The b row of $_3T'$ is obtained in the same way as for $_2T'$ except that now each element of row a is divided by the corresponding diagonal element of the $_3g$ matrix of Table 6. The largest value in row b of Table 7 is .3005 for variable 4. Therefore, the third or cth predictor is variable 4. By means of Equation 31 of Part II we have as our index of absolute prediction efficiency for the first three selected predictors

$$\lambda_3 = \lambda_2 + {}_3\Delta_4 = 2.1814 + .3005 = 2.4819.$$

11. Steps 6 through 10 are repeated to select the fourth predictor.

a. The $_3G_4'$ vector in the third row of Table 5 is obtained from the fourth row of the $_3g$ matrix in Table 6 and its fourth

TABLE 7

THE $_3T'$ MATRIX

	1	2	3	4	5	6	7	8
1	.1598	.0872	.1675	.2075	0	.0388	.0484	0
2	.0318	.1733	.0831	.0733	0	.0717	.1452	0
3	.1053	.1507	.0946	.1571	0	.0331	.0943	0
4	.1843	.1007	.1323	.1465	0	.1096	.0005	0
5	.1477	.1524	.2120	.1328	0	.1139	.0050	0
6	.0767	.0972	.0840	.1132	0	.0344	.0426	0
7	.1392	.0516	.1359	.1851	0	.0366	−.0400	0
8	.0513	.1773	.1385	.0320	0	.0879	.1798	0
9	.0923	.0656	.0843	.1390	0	.0385	.0364	0
10	.0835	.1577	.0601	.0949	0	.0407	.0633	0
Check	1.0719	1.2137	1.1925	1.2812	0	.6053	.5755	0
Σ	1.0719	1.2137	1.1923	1.2814	0	.6052	.5755	0
a	.1368	.1664	.1618	.1886	0	.0461	.0734	0
b	.2200	.2528	.2373	.3005	0	.0570	.0893	0

TABLE 8

THE $_4g$ MATRIX

	1	2	3	4	5	6	7	8
1	.4328							
2	.1093	.6054						
3	.1743	.1964	.6020					
4	0	0	0	0				
5	0	0	0	0	0			
6	.1355	.0786	.1103	0	0	.7730		
7	−.0530	.2132	−.0039	0	0	.0675	.8210	
8	0	0	0	0	0	0	0	0
Check	.7988	1.2029	1.0791	0	0	1.1649	1.0447	0
Σ	.7989	1.2029	1.0791	0	0	1.1649	1.0448	0

diagonal element.

b. The $_4g$ matrix of Table 8 is obtained from the $_3g$ matrix of Table 6, its fourth column and the third row of Table 5.

c. The $_4T'$ matrix of Table 9 is obtained from the $_3T'$ matrix of Table 7, its fourth column and the third row of Table 5.

d. The *a* row of Table 9 is simply the sums of squares of column elements of the $_4T'$ matrix.

e. The $_4\Delta_k$ values in row *b* of Table 9 are obtained from the *a* row of Table 9 and the diagonal elements of Table 8.

The procedure has been repeated until all eight predictor variables have been selected. The remaining rows of Table 5 give the successive G' vectors in the order in which the variables were selected. Tables 10 through 17 give alternately the remaining $_ig$ and $_iT'$ matrices.

For convenient reference the successive indices of absolute prediction efficiency are summarized below.

$\lambda_1 = {}_1\Delta_6 = 1.6771$

$\lambda_2 = \lambda_1 + {}_2\Delta_3 = 1.6771 + .5043 = 2.1814$

$\lambda_3 = \lambda_2 + {}_3\Delta_4 = 2.1814 + .3005 = 2.4819$

$\lambda_4 = \lambda_3 + {}_4\Delta_2 = 2.4819 + .1673 = 2.6492$

$\lambda_5 = \lambda_4 + {}_5\Delta_5 = 2.6492 + .0726 = 2.7218$

$\lambda_6 = \lambda_5 + {}_6\Delta_7 = 2.7218 + .0504 = 2.7722$

$\lambda_7 = \lambda_6 + {}_7\Delta_1 = 2.7722 + .0227 = 2.7949$

$\lambda_8 = \lambda_7 + {}_8\Delta_8 = 2.7949 + .0119 = 2.8068$

C. Computations for the Regression Weights

In solving for the regression vectors we

TABLE 9

THE $_4T'$ MATRIX

	1	2	3	4	5	6	7	8
1	.0459	.0270	.0936	0	0	−.0104	.0563	0
2	−.0084	.1521	.0570	0	0	.0543	.1480	0
3	.0191	.1052	.0386	0	0	−.0042	.1003	0
4	.1039	.0582	.0801	0	0	.0748	.0061	0
5	.0748	.1139	.1647	0	0	.0824	.0100	0
6	.0146	.0644	.0437	0	0	.0075	.0469	0
7	.0376	−.0020	.0699	0	0	−.0073	−.0330	0
8	.0337	.1680	.1271	0	0	.0803	.1810	0
9	.0160	.0253	.0348	0	0	.0055	.0417	0
10	.0314	.1302	.0263	0	0	.0182	.0669	0
Check	.3687	.8424	.7357	0	0	.3011	.6241	0
Σ	.3686	.8423	.7358	0	0	.3011	.6242	0
a	.0229	.1013	.0719	0	0	.0224	.0775	0
b	.0529	.1673	.1194	0	0	.0390	.0943	0

TABLE 10

THE $_{8}g$ MATRIX

	1	2	3	4	5	6	7	8
1	.4131							
2	0	0						
3	.1388	0	.5383					
4	0	0	0	0				
5	0	0	0	0	0			
6	.1213	0	.0848	0	0	.7628		
7	−.0915	0	−.0731	0	0	.0398	.7459	
8	0	0	0	0	0	0	0	0
Check	.5818	0	.6889	0	0	1.0088	.6211	0
Σ	.5817	0	.6888	0	0	1.0087	.6211	0

TABLE 11

THE $_{8}T'$ MATRIX

	1	2	3	4	5	6	7	8
1	.0410	0	.0848	0	0	−.0139	.0468	0
2	−.0359	0	.0077	0	0	.0346	.0944	0
3	.0001	0	.0045	0	0	−.0179	.0632	0
4	.0934	0	.0612	0	0	.0672	−.0144	0
5	.0542	0	.1278	0	0	.0676	−.0301	0
6	.0030	0	.0228	0	0	−.0009	.0242	0
7	.0380	0	.0705	0	0	−.0070	−.0323	0
8	.0034	0	.0726	0	0	.0585	.1218	0
9	.0114	0	.0266	0	0	.0022	.0328	0
10	.0079	0	−.0159	0	0	.0013	.0210	0
Check	.2167	0	.4626	0	0	.1918	.3275	0
Σ	.2165	0	.4626	0	0	.1917	.3274	0
a	.0163	0	.0391	0	0	.0143	.0342	0
b	.0395	0	.0726	0	0	.0187	.0458	0

TABLE 12

THE $_{8}g$ MATRIX

	1	2	3	4	5	6	7	8
1	.3773							
2	0	0						
3	0	0	0					
4	0	0	0	0				
5	0	0	0	0	0			
6	.0994	0	0	0	0	.7494		
7	−.0727	0	0	0	0	.0513	.7360	
8	0	0	0	0	0	0	0	0
Check	.4041	0	0	0	0	.9002	.7146	0
Σ	.4040	0	0	0	0	.9001	.7146	0

TABLE 13

THE $_{8}T'$ MATRIX

	1	2	3	4	5	6	7	8
1	.0192	0	0	0	0	−.0273	.0583	0
2	−.0379	0	0	0	0	.0334	.0954	0
3	−.0011	0	0	0	0	−.0186	.0638	0
4	.0776	0	0	0	0	.0576	−.0061	0
5	.0213	0	0	0	0	.0475	−.0127	0
6	−.0029	0	0	0	0	−.0045	.0273	0
7	.0198	0	0	0	0	−.0181	−.0227	0
8	−.0153	0	0	0	0	.0471	.1317	0
9	.0045	0	0	0	0	−.0020	.0364	0
10	.0120	0	0	0	0	.0038	.0188	0
Check	.0972	0	0	0	0	.1188	.3902	0
Σ	.0972	0	0	0	0	.1189	.3902	0
a	.0091	0	0	0	0	.0104	.0371	0
b	.0241	0	0	0	0	.0139	.0504	0

TABLE 14

THE $_7g$ MATRIX

	1	2	3	4	5	6	7	8
1	.3701							
2	0	0						
3	0	0	0					
4	0	0	0	0				
5	0	0	0	0	0			
6	.1045	0	0	0	0	.7458		
7	0	0	0	0	0	0	0	
8	0	0	0	0	0	0	0	0
Check	.4746	0	0	0	0	.8503	0	0
Σ	.4746	0	0	0	0	.8503	0	0

TABLE 16

THE $_8g$ MATRIX

	1	2	3	4	5	6	7	8
1	0							
2	0	0						
3	0	0	0					
4	0	0	0	0				
5	0	0	0	0	0			
6	0	0	0	0	0	.7163		
7	0	0	0	0	0	0	0	
8	0	0	0	0	0	0	0	0
Check	0	0	0	0	0	.7163	0	0
Σ	0	0	0	0	0	.7163	0	0

shall assume as in the case of Differential Prediction that only the first four selected predictors are to be used. The successive steps to be followed are numbered to correspond to the steps in Part II, Section *B*.

1. We first prepare the *L* matrix of Equation 32, Part II from the appropriate entries in Table 5. This gives us Table 18. The first row of Table 18 is obtained from the first row of Table 5. It includes, in the order of selection, only those elements corresponding to the selected variables. Note that the first entry, the one corresponding to the unit element, is zero. The other elements are all copied with opposite sign from the first row of Table 5.

TABLE 15

THE $_7T'$ MATRIX

	1	2	3	4	5	6	7	8
1	.0250	0	0	0	0	−.0314	0	0
2	−.0285	0	0	0	0	.0268	0	0
3	.0052	0	0	0	0	−.0230	0	0
4	.0770	0	0	0	0	.0580	0	0
5	.0200	0	0	0	0	.0484	0	0
6	−.0002	0	0	0	0	−.0064	0	0
7	.0176	0	0	0	0	−.0165	0	0
8	−.0023	0	0	0	0	.0379	0	0
9	.0081	0	0	0	0	−.0045	0	0
10	.0139	0	0	0	0	.0025	0	0
Check	.1358	0	0	0	0	.0917	0	0
Σ	.1358	0	0	0	0	.0918	0	0
a	.0084	0	0	0	0	.0097	0	0
b	.0227	0	0	0	0	.0130	0	0

TABLE 17

THE $_8T'$ MATRIX

	1	2	3	4	5	6	7	8
1	0	0	0	0	0	−.0385	0	0
2	0	0	0	0	0	.0348	0	0
3	0	0	0	0	0	−.0245	0	0
4	0	0	0	0	0	.0363	0	0
5	0	0	0	0	0	.0428	0	0
6	0	0	0	0	0	−.0063	0	0
7	0	0	0	0	0	−.0215	0	0
8	0	0	0	0	0	.0385	0	0
9	0	0	0	0	0	−.0068	0	0
10	0	0	0	0	0	−.0014	0	0
Check	0	0	0	0	0	.0534	0	0
Σ	0	0	0	0	0	.0534	0	0
a	0	0	0	0	0	.0085	0	0
b	0	0	0	0	0	.0119	0	0

The first two entries in the second row are zero. The remaining elements are copied, with sign reversed, from the elements in the second row of Table 5 corresponding in order of selection to all but the first two selected predictors. The remaining rows are similarly obtained from the corresponding rows of Table 5.

TABLE 18

THE *L* MATRIX

	5	8	4	2	Σ
5	0	−.3185	−.5936	−.5801	−1.4922
8	0	0	−.1490	−.0776	−.2266
4	0	0	0	−.2898	−.2898
2	0	0	0	0	0

TABLE 19

THE F' MATRIX WITH CHECKS

	5	8	4	2	Σ	Checks*	
						$1-L1$	$F'(g_{(i)}1)$
5	1.0000	0	0	0	1.0000	2.4922	2.4922
8	−.3544	1.1128	0	0	.7584	1.2266	1.2266
4	−.8702	−.2374	1.5934	0	.4858	1.2898	1.2901
2	−.6559	−.0569	−.4787	1.6518	.4603	1.0000	1.0001
Σ	−.8805	.8185	1.1147	1.6518	2.7045		

* See text.

TABLE 20

THE $g_{(i)}$ MATRIX

	5	8	4	2	Σ
5	1.0000	.3185	.5936	.5801	2.4922
8	.3185	1.0000	.3230	.2545	1.8960
4	.5936	.3230	1.0000	.5366	2.4532
2	.5801	.2545	.5366	1.0000	2.3712

The rows of the L matrix are summed and the sums are entered in the corresponding rows of the Σ column at the right of the L matrix.

2. Next we prepare the F' matrix of Equation 33, Part II as shown in Table 19. The diagonal elements of this matrix are copied from the left-hand column of Table 5. These are the elements indicated by Equations 34 of Part II. To get the first element in the second row of Table 19 we use Equation 35 of Part II. This gives

$$-.3544 = -.3185 \times 1.1128.$$

The second and first elements in the third row of Table 19 are, respectively, by Equations 36 of Part II

$$-.2374 = (-.1490) \times 1.5934,$$

$$-.8702 = (-.3185) \times (-.2374) - .5936 \times 1.5934.$$

Similarly by means of Equations 37 of Part II the third, second, and first elements of the fourth row of Table 19 are computed.

The columns of F' are summed for subsequent checking purposes.

The elements in the second column from the right in Table 19 are obtained by subtracting from unity the value obtained for the corresponding row sum in Table 18.

3. To check the F' matrix we first compute the sums of columns (or rows) of the matrix of intercorrelations of the selected predictors. These sums may be calculated directly from the $_1g$ matrix of Table 1 without actually copying out the matrix for the selected predictors. To clarify the procedure, however, the matrix of intercorrelations for the selected predictors is given in Table 20 together with the row sums.

It will be recalled that the entries in the second column from the right in Table 19

TABLE 21

THE I MATRIX

	1	2	3	4	5	6	7	8	9	10	Σ
5	.3950	.4966	.4329	.4184	.3855	.3836	.2870	.4156	.3799	.4656	4.0601
8	.2506	.1309	.2342	.3094	.1191	.1963	.1985	.0941	.2177	.2727	2.0235
4	.2075	.0733	.1571	.1465	.1328	.1132	.1851	.0320	.1390	.0949	1.2814
2	.0270	.1521	.1052	.0582	.1139	.0644	−.0020	.1680	.0253	.1302	0.8423

TABLE 22

THE β MATRIX

	1	2	3	4	5	6	7	8	9	10
5	.1079	.2867	.1442	.1431	.1530	.1733	.0569	.2442	.1652	.2010
8	.2281	.1196	.2173	.3062	.0945	.1879	.1771	.0876	.2078	.2735
4	.3177	.0440	.2000	.2056	.1571	.1495	.2959	−.0294	.2094	.0889
2	.0446	.2512	.1738	.0961	.1881	.1064	−.0033	.2775	.0418	.2151
Check	.6983	.7015	.7353	.7510	.5928	.6171	.5265	.5798	.6242	.7784
Σ	.6983	.7015	.7353	.7510	.5927	.6171	.5266	.5799	.6242	.7785

are unity minus the corresponding row sum in Table 18. These are the values given to the right of Equations 38 in Part II. The left sides of Equation 38 or 39 are obtained by summing the products of the row sums of Table 20 by the corresponding elements of a row of the F' matrix. For example, the check for the last row of F' is

$$-.6559\ (2.4922)-.0569\ (1.8960)$$
$$-.4787\ (2.4532)+1.6518\ (2.3712)$$
$$=1.0001.$$

The entries to the extreme right in Table 19 are these product summations and should check within rounding errors with the entries at their left.

4. The rows of the t matrix given by Equation 40 of Part II are taken from the appropriate columns of the ${}_iT'$ matrices. This matrix is given in Table 21.

The first row is the ath or fifth column of the ${}_1T'$ matrix of Table 2.

The second row is the bth or eighth column of the ${}_2T'$ matrix, and so on.

5. By means of Equation 41 or 42 of Part II the matrix of regression vectors β in Table 22 is obtained from Tables 19 and 21.

The first column of Table 22 is obtained from Table 19 and the first column of Table 21 as follows:

The first element of the column is the sum of products of corresponding elements in the first column of Table 19 and the first column of Table 21. The computations for the first element are

$$1.000\ (.3950)-.3544\ (.2506)-.8702$$
$$(.2075)-.6559\ (.0270)=.1079.$$

The second element in the column is the sum of products of corresponding elements in the second column of Table 19 and the first column of Table 21. Each succeeding element in the column is the sum of products of corresponding elements in the corresponding column of Table 19 and the first column of Table 21. The element in the check position for the first column of Table 22 is the product of corresponding elements in the summation column of Table 19 and the first column of Table 21. The element in the summation position in Table 22 is the sum of all elements in the column exclusive of the check entry. The check and summation entry should be the same within limits of rounding error.

The elements in the second column of Table 22 are obtained in the same way as the first except now the second column of Table 21 is used instead of the first. The remaining columns are obtained by using the corresponding column of Table 21 with Table 19.

D. Computations for the Multiple Correlations

The conventional method for solving for the squares of the multiple correlations is indicated by Equation 43 or 44 of Part II.

TABLE 23

THE $_1T_{(4)}$ MATRIX

	1	2	3	4	5	6	7	8	9	10	Σ
5	.3950	.4966	.4329	.4184	.3855	.3836	.2870	.4156	.3799	.4656	4.0601
8	.3764	.2891	.3721	.4427	.2419	.3185	.2899	.2265	.3387	.4210	3.3168
4	.4793	.3876	.4490	.4410	.3793	.3701	.3851	.2928	.3969	.4119	3.9930
2	.3357	.4716	.4200	.3674	.3852	.3349	.2335	.4257	.3029	.4490	3.7259
R^2	.2957	.3125	.3060	.3214	.2139	.2173	.1808	.2308	.2289	.3419	2.6492
R	.5438	.5590	.5532	.5669	.4625	.4662	.4252	.4805	.4784	.5847	5.1204

1. For computational convenience it is well to copy in row form the columns from Table 2 corresponding to the selected predictors. These should be copied in the order selected as shown in Table 23 and are designated as the $_1T_{(4)}$ matrix. The sums should also be copied and the rows totaled to check the copying.

2. The squares of the multiple correlations are then computed by getting the sums of products of corresponding elements of corresponding columns of Tables 22 and 23. These are entered in the row immediately below Table 23. The first entry in this row will be obtained from the first columns of Tables 22 and 23 as follows:

$$.2957 = .3950\ (.1079) + .3764\ (.2281) + .4793\ (.3177) + .3357\ (.0446).$$

3. The check for the R^2's is given by

$$\lambda_i = \Sigma R_k^2,$$

where k refers to the criterion variable. In this case λ_i is λ_4. We have therefore,

$$\lambda_4 = 2.6492 = \Sigma R^2 = 2.6492.$$

4. The last row under Table 23 is merely the square roots of the elements in the row above, and gives the multiple correlations rather than their squares.

IV. Mathematical Derivations

A. The Index of Absolute Prediction Efficiency

The index of absolute prediction efficiency employed in the methods outlined in the previous sections was based on the assumption that we wish to obtain from the predictors the best estimates in the least-square sense for all of the criterion measures. We assume standard measures for both predictor and criterion scores. We let:

M = number of cases,
n = number of predictors,
N = number of criteria,
X = the $(M \times n)$ matrix of predictor measures,
Y = the $(M \times N)$ matrix of criterion measures,
Z = the $(M \times N)$ matrix of best least-square estimates of Y obtained from X,
β = the $(n \times N)$ matrix of least-square regression vectors for estimating Y from X,
$_1g$ = the $(n \times n)$ matrix of intercorrelations of the complete set of potential predictors from which the subset is to be selected,
$_1T$ = the $(n \times N)$ matrix of validity coefficients, i.e., the correlations of the predictors with the criteria,
r = the $(N \times N)$ matrix of intercorrelations of the criteria,
C = $(N \times N)$ matrix of covariances of the predicted criterion measures.

Because of the above definitions,

$$\frac{X'X}{M} = {}_1g, \qquad [1]$$

$$\frac{X'Y}{M} = {}_1T, \qquad [2]$$

$$\frac{Y'Y}{M} = r, \qquad [3]$$

$$\frac{Z'Z}{M} = C, \qquad [4]$$

$$X\beta = Z. \qquad [5]$$

We now consider the residual matrix

$$E = (Y - X\beta). \qquad [6]$$

It is the trace of $E'E$ which we wish to minimize.

The solution for β in Equation 6 which minimizes the trace of $E'E$ is well known to be

$$\beta = (X'X)^{-1}X'Y. \qquad [7]$$

Substituting 1 and 2 in 7 gives

$$\beta = {}_1g^{-1}\,{}_1T. \qquad [8]$$

Substituting 8 in 6, gives

$$E = (Y - X\,{}_1g^{-1}\,{}_1T). \qquad [9]$$

From 9 we have

$$E'E = (Y'Y - {}_1T'\,{}_1g^{-1}X'Y - Y'X\,{}_1g^{-1}\,{}_1T + {}_1T'\,{}_1g^{-1}X'X\,{}_1g^{-1}\,{}_1T), \qquad [10]$$

or substituting from 1, 2, and 3 in 10 gives

$$\frac{E'E}{M} = (r - {}_1T'\,{}_1g^{-1}\,{}_1T). \qquad [11]$$

Obviously if we minimize the trace of $E'E/M$ we also minimize the trace of $E'E$. Furthermore, the diagonal elements of r are all unity, hence the trace of r is N or a constant. Therefore, we may take as an index of multiple absolute prediction efficiency the trace of the right-hand term on the right of Equation 11, and designating this by λ we have

$$\lambda = tr({}_1T'\,{}_1g^{-1}\,{}_1T). \qquad [12]$$

But it can readily be shown from 1, 4, 5, and 8 that

$$C = {}_1T'\,{}_1g^{-1}\,{}_1T. \qquad [13]$$

According to 12 and 13 therefore, λ is simply the sum of the diagonal elements of C or the sum of the variances of the predicted criterion variables. But it is well known that the diagonal elements of the right side of Equation 13 are simply the squares of the multiple correlations of the criteria with the predictors.

B. The Predictor Selection Formulas

It should be clear then that for any given set of predictor and criterion variables the matrix of vectors β which will yield the highest index of multiple absolute prediction efficiency is given by Equation 8 and is the one which will yield the most accurate estimates, in the least-square sense, of the separate criterion measures. As shown in Differential Prediction this is also the matrix which will yield the highest index of multiple differential prediction efficiency. However, the problem is to select from a larger set of predictors that subset which will yield a λ as given in Equation 12 as large as, or larger than, any other subset of equal size. As in the case of multiple differential prediction, it seems probable that for a given sample of data the only way that the subset of specified size having maximum λ could be found is actually to calculate the λ's for all possible sets of the specified size. However, suppose we select first the single predictor with the largest λ; to this we add the predictor which yields the greatest increment to the λ; and we continue in this way to select one predictor at a time until the subset of desired size has been selected. As in the selection procedure for the differential prediction battery we shall assume that the selection of one predictor at a time will yield a subset whose λ is sufficiently close for practical purposes to that of the subset of equal size with the largest λ.

We begin, therefore, by deriving recursion formulas based on Equation 12 but involving a total number of predictor variables smaller than the total number available. Let us assume that we have already selected i predictors and calculated the corresponding λ, say λ_i, and that we wish to find which of the remaining $n-i$ predictors will yield the largest increment ${}_{(i+1)}\Delta$ to λ_i.

We let:

$g_{(i)}$ = the $(i \times i)$ matrix of intercorrelations of i selected predictors arranged in the order of selection,

$T_{(i)}$ = the $(i \times N)$ matrix of correlations of the i predictors with the N criteria arranged in the order of selection,

$g_{(i+1_k)}$ = the matrix of intercorrelations of the i selected predictors and one unselected predictor k,

$T_{(i+1_k)}$ = the matrix of correlations of i selected predictors and one unselected predictor k with the N criteria,

$g_{(i)k}$ = the column vector of correlations of the i selected predictors with unselected predictor k,

$T_{k.}$ = the column vector of correlations of predictor k with the N criteria,

$C_{(i)}$ = the covariance matrix of predicted criterion scores derived from the i selected predictors,

$C_{(i+1_k)}$ = the covariance matrix of predicted criterion scores derived from the i selected predictors and an unselected predictor k.

According to the above definitions:

$$g_{(i+1_k)} = \begin{pmatrix} g_{(i)} & g_{(i)k} \\ g_{(i)k}' & 1 \end{pmatrix}, \qquad [14]$$

$$T_{(i+1_k)} = \begin{pmatrix} T_{(i)} \\ T_{k.}' \end{pmatrix}. \qquad [15]$$

Using Equation 13 we have:

$$C_{(i)} = T_{(i)}' g_{(i)}^{-1} T_{(i)}, \qquad [16]$$

$$C_{(i+1_k)} = T_{(i+1_k)}' g_{(i+1_k)}^{-1} T_{(i+1_k)}. \qquad [17]$$

Suppose now we define[2]

$$\beta_{(i)k} = g_{(i)}^{-1} g_{(i)k}. \qquad [18]$$

Then we can prove that

$$g_{(i+1_k)}^{-1} = \begin{pmatrix} g_{(i)}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{\begin{pmatrix} -\beta_{(i)k} \\ 1 \end{pmatrix} (-\beta_{(i)k}', 1)}{1 - g_{(i)k}' \beta_{(i)k}} \qquad [19]$$

Substituting Equations 15 and 19 in 17 gives

$$C_{(i+1_k)} = T_{(i)}' g_{(i)}^{-1} T_{(i)} + \frac{(T_{k.} - T_{(i)}' \beta_{(i)k})(T_{k.}' - \beta_{(i)k}' T_{(i)})}{1 - g_{(i)k}' \beta_{(i)k}}. \qquad [20]$$

(In the following development, the distinction between the pre- and postscripts should be constantly borne in mind. The prescript refers to the number of predictors selected [that is, the number of iterations] and has no reference to the order of the matrix. The parenthesized postscript, however, refers to the number of predictor variables represented in the matrix, and hence affects the order of the T and g matrices to which it is attached.)

Suppose now we let:

$${}_{i}T_{k.} = T_{k.} - T_{(i)}' \beta_{(i)k}, \qquad [21]$$

and

$${}_{i}g_{kk} = 1 - g_{(i)k}' \beta_{(i)k}. \qquad [22]$$

Substituting Equations 16, 21, and 22 in 20 gives

$$C_{(i+1_k)} = C_{(i)} + \frac{{}_{i}T_{k.} {}_{i}T_{k.}'}{{}_{i}g_{kk}}. \qquad [23]$$

[2] Not to be confused with β defined in Equation 8.

Now if we let $D_{c(i)}$ and $D_{c(i+1k)}$ be the diagonal matrices of the diagonal elements of $C_{(i)}$ and $C_{(i+1k)}$ respectively, and $\lambda_{(i)}$ and $\lambda_{(i+1k)}$ the corresponding indices of absolute prediction efficiency, then because of Equations 12 and 13

$$\lambda_{(i)} = 1' D_{c(i)} 1, \qquad [24]$$

$$\lambda_{(i+1k)} = 1' D_{c(i+1k)} 1, \qquad [25]$$

and if we let

$$_i\Delta_k = \lambda_{(i+1k)} - \lambda_{(i)}, \qquad [26]$$

we have because of Equations 23, 24, and 25

$$_i\Delta_k = \frac{{}_iT_{k.}' \; {}_iT_{k.}}{{}_ig_{kk}}. \qquad [27]$$

Equation 27 corresponds to Equation 25 of Part II. This equation indicates that the crucial requirement for the selection procedure is that recursion formulas be available for the computation of the $_i\Delta_k$ increments.

In order to develop the recursion formulas, first let us consider the equation analogous to Equation 21 assuming that i predictors plus predictor k have been selected and that we wish to investigate the unselected predictor p. We would have,

$$_{(i+1)}T_{p.} = {}_iT_{p.} - {}_iT_{(i+1k)}'\beta_{(i+1k)p}. \qquad [28]$$

Suppose now we let g_{kp} be the correlation between predictors k and p, and we define

$$_ig_{kp} = g_{kp} - \beta_{(i)k}' g_{(i)p}, \qquad [29]$$

and

$$_iG_{kp} = \frac{{}_ig_{kp}}{{}_ig_{kk}}. \qquad [30]$$

Then as was demonstrated by means of Equations 62 through 71, Part IV of Differential Prediction, we can rewrite Equation 28 as

$$_{(i+1)}T_{p.} = {}_iT_{p.} - {}_iT_{k.}\,{}_iG_{kp}. \qquad [31]$$

If now we assume that predictor k was the $(i+1)$th variable selected, let $k=w$, and write Equation 31 in matrix form to include all values of p from 1 to n we have

$$_{(i+1)}T = {}_iT - {}_iG_w\,{}_iT_{w.}', \qquad [32]$$

where $_iT_{w.}'$ is the wth row from $_iT$ and $_iG_w$ is a column vector. Equation 32 corresponds to Equation 28 of Part II.

Our next step is to get recursion formulas for the $_ig_{kp}$ values given by Equation 29.

Suppose we have selected w as the $(i+1)$th predictor. We wish to find all $_{(i+1)}g_{kp}$ values. From Equation 29 we write

$$_{(i+1)}g_{kp} = g_{kp} - \beta_{(i+1w)k}'\, g_{(i+1w)p}. \qquad [33]$$

Using the methods of Equations 74 through 76 of Differential Prediction we rewrite Equation 33 as

$$_{(i+1)}g_{kp} = {}_ig_{kp} - \frac{{}_ig_{wk}\,{}_ig_{wp}}{{}_ig_{ww}}. \qquad [34]$$

We let:

$_{(i+1)}g$ = the $(n \times n)$ matrix for all values of k and p on the left of Equation 34,

$_ig$ = the $(n \times n)$ matrix for all values of k and p in the first term on the right of Equation 34,

$_ig_{.w}$ = the wth column vector of $_ig$.

Analogous to Equation 30 we define

$$_iG_w = \frac{{}_ig_{.w}}{{}_ig_{ww}}. \qquad [35]$$

Using Equation 35 and the definitions just given, and assuming predictor w as the ith predictor selected, we rewrite Equation 34 in matrix form as

$$_{(i+1)}g = {}_ig - {}_ig_{.w}\,{}_iG_w'. \qquad [36]$$

Equation 35 corresponds to Equation 26 of Part II while Equation 36 corresponds to Equation 27 of Part II.

The selection process begins with the

${}_1T$ matrix of validity coefficients. By means of Equation 27 for the case of $i=1$ we compute ${}_1\Delta_k$ for all values of k. Since all diagonal values of ${}_1g$ are unity only the numerator term of Equation 27 is required. The predictor with the highest ${}_1\Delta$ is the first one selected.

By means of Equation 36 a ${}_2g$ matrix is computed where w is the first or ath variable selected. Since all diagonal elements of ${}_1g$ are unity, ${}_1G_w'$ given by Equation 35 is the same as ${}_1g_{.w}$. This operation is identical to the calculation of the ${}_2g$ matrix described in Differential Prediction.

A ${}_2T$ matrix is now computed by means of Equation 32 where $i=1$ and w is the first variable selected. By means of Equation 27 a second predictor is chosen. The routine continues until the desired number of predictors has been selected.

C. *The Matrix of Regression Vectors*

The formulas for computing the matrix of regression vectors are exactly the same as in the case of Differential Prediction. We shall, however, indicate and prove a superior checking procedure for one of the formulas. First we let ${}_i\gamma_w$ be a vector of those elements from ${}_iG_w$ of Equation 35 corresponding to the selected predictors and arranged in the order selected. Then $\gamma_{(i)}$ is defined as a matrix whose rows are the ${}_i\gamma_w'$ vectors, where w goes from a to w. We also let

$$D_w=\begin{pmatrix} {}_1g_{aa} & 0 & \cdots & 0 \\ 0 & {}_2g_{bb} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & {}_ig_{ww} \end{pmatrix}. \qquad [37]$$

Then it can be shown, because of the method by which $\gamma_{(i)}$ was computed, that

$$g_{(i)}=\gamma_{(i)}'D_w\gamma_{(i)}. \qquad [38]$$

We now define a matrix F by

$$F=(D_w\gamma_{(i)})^{-1}. \qquad [39]$$

This is the transpose of the matrix indicated by Equation 33 of Part II. The method of computation is precisely the same as outlined in Differential Prediction. However, a superior method for checking the computations of F' is given by Equations 38 and 39 of Part II. Equation 39 of Part II is

$$F'(g_{(i)}1)=1-L1, \qquad [40]$$

where L is defined by the equation

$$L=I-\gamma_{(i)}. \qquad [41]$$

We can readily prove that Equation 40 is an identity. From Equation 39 we have

$$F'=D_w^{-1}\gamma_{(i)}'^{-1}. \qquad [42]$$

Substituting Equations 38, 41, and 42 in 40 we have

$$(D_w^{-1}\gamma_{(i)}'^{-1})(\gamma_{(i)}'D_w\gamma_{(i)})1 = 1-(I-\gamma_{(i)})1, \qquad [43]$$

or

$$\gamma_{(i)}1=\gamma_{(i)}1. \qquad [44]$$

The solution for the regression vectors, once the F matrix has been computed and checked, is indicated by Equation 41 or 42 of Part II, and proved in Differential Prediction.

D. *The Multiple Correlation Formulas*

The method for computing the squares of the multiple correlations indicated in Equation 43, Part II, Section *C*, is the same as one of the methods indicated in Differential Prediction. This method is well known and therefore no proof of the method was given in Differential Prediction nor will it be given here. However, the check for the computations indicated in Part II, Section *C*, may not be obvious. We need only recall that the definition of λ given by Equation 12 for any set of predictors is the sum of the diagonal elements of the matrix in Equation 13.

From Equations 8 and 13

$$C = {}_1T'\beta. \qquad [45]$$

But the kth diagonal element of Equation 45 is given by

$$C_{kk} = {}_1T_{.k}'\beta_{.k}. \qquad [46]$$

The right side of Equation 46 is the same as the right side of Equation 44 of Part II. We have, therefore, from Equations 12, 13, 26, and 46

$$\lambda_i = \Sigma R_k^2, \qquad [47]$$

where the R_k^2 values are the squares of the multiple correlations of the i selected predictors with the various criteria.

V. Summary

This report presented (*a*) a short discussion of the multiple absolute prediction problem as contrasted with that of differential prediction, (*b*) a detailed description of the computational methods developed for selecting the absolute predictors and solving for the regression vectors and multiple correlations. It also includes (*c*) a numerical illustration of the computational procedure, and (*d*) the mathematical derivation of the formulas developed.

The multiple absolute prediction problem, as distinguished from that of differential prediction was stated as the problem of selecting from a battery of potential predictors that subset of specified size which will have the highest prediction efficiency for all the criterion variables, irrespective of how well it differentiates among them. The index of the prediction efficiency of the selected battery was defined as the sum of the variances of the predicted criteria irrespective of their covariances.

In both differential and absolute prediction, the end result is a set of predictions of success in each of the criterion activities. The weights to be used in both cases are found to be the conventional least-square regression weights, the essential difference between the two cases being that the sets of selected predictors are not in general the same.

Analogous to the technique presented in Differential Prediction (1), an iterative procedure was developed for selecting, progressively, the predictor which, when combined with the previously selected set, yields the highest index of absolute prediction efficiency. The process is continued until the prespecified number of predictors has been so selected. It is assumed that this procedure yields an index of absolute prediction efficiency sufficiently close, for practical purposes, to that of the particular set of specified size which, for the particular data, would yield the highest index of absolute prediction efficiency.

The numerical example presented in the computational section consists of operations on the same matrices as those used in the computational section of Differential Prediction, namely, a matrix of eight predictor variables and a matrix of ten criterion variables. The reader may thereby compare the results obtained for the two types of multiple prediction.

The mathematical section comprises the development of the rationale for the selection procedure, and of formulas for obtaining the matrix of beta weights and the multiple correlations of the selected predictors with the criteria. This development eliminates the necessity for computing the inverse of the predictor intercorrelations.

Reference

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